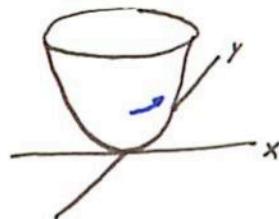


# Nonlinear Hamiltonian PDE

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$$H: \mathbb{R}_{x,y}^2 \mapsto \mathbb{R} \quad \text{e.g. } H(x,y) = x^2 + y^2.$$

$\mathbb{R} \ni t \mapsto (x(t), y(t))$  is a flow on  $\mathbb{R}^2$  s.t.



$$\textcircled{H} \quad \begin{cases} \dot{x} = H_y \\ \dot{y} = -H_x \end{cases} \quad \text{Hamilton's Equations} \quad \bullet = \frac{d}{dt}.$$

This is a special case of the general 1st order ODE in 2 variables:

$$\begin{cases} \dot{x} = F(x, y) \\ \dot{y} = G(x, y) \end{cases}$$

Along the Hamilton flow  $t \mapsto (x(t), y(t))$ ,

$$\frac{d}{dt} H(x(t), y(t)) = H_x \dot{x} + H_y \dot{y} = H_x H_y + H_y (-H_x) = 0.$$

The flow moves along level sets of  $H$ .

Reexpressions of  $\textcircled{H}$ .

$$\underbrace{\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbb{H}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{:= \mathcal{J}} \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

$$\boxed{\dot{w} = \mathcal{J} \nabla H.}$$

Vector representation  
of  $\textcircled{H}$

$$\underline{x} \in \mathbb{R}^2.$$

- $\mathcal{J} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ .  
So  $\mathcal{J}$  rotates vectors by  $-\frac{\pi}{2}$  radians.
- Motion is  $\perp \nabla H$   
so it stays in level set.
- $\mathcal{J}^2 = -\mathbb{I}$   
so some connection to  $\textcircled{C}$ ?

②

$$z = x + iy$$

chain rule

$$\frac{\partial}{\partial z} = \frac{\partial z}{\partial x} \frac{\partial}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial}{\partial y}$$

$$\bar{z} = x - iy.$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial \bar{z}}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \bar{z}}{\partial y} \frac{\partial}{\partial y}$$

algebra:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

$$\dot{z} = \dot{x} + i \dot{y} = H_y - i H_x = -i \{ H_x + i H_y \} = -i 2 H_{\bar{z}}.$$

$$\dot{z} = -i 2 H_{\bar{z}}.$$

$$\boxed{\dot{z} = -i H_{\bar{z}}}$$

Complex Representation  
of  $\mathbb{H}$

Example

$$H(x, y) = x^2 + y^2 \quad \text{or}$$

$$H(z, \bar{z}) = |z|^2 = z \bar{z}.$$

$$\dot{x} = 2y$$

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{z} = -i z.$$

$$\dot{y} = -2x$$

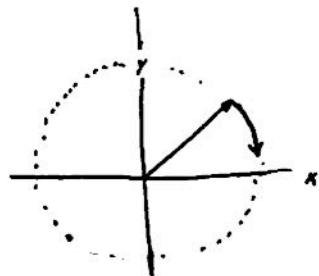
(vector)

(standard)

(complex)

$$\text{solution: } z_0 \mapsto z(t)$$

$$z(t) = e^{-it} z_0.$$



Think of these dynamics  
sitting "above" a point.

③

(3)

### Jacked up Example

$$H: \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

$j=1, \dots, n$

$$\begin{array}{ccc} \mathbb{R}^{2n} & & \{(x_1, y_1), \dots, (x_n, y_n)\} \\ \downarrow & & (\underline{x}, \underline{y}) \in \mathbb{R}^n \times \mathbb{R}^n \\ \mathbb{C}^n & & \text{etc.} \quad \underline{w} = (\underline{x}, \underline{y}). \end{array}$$

$$\begin{cases} \dot{x}_j = H y_j \\ \dot{y}_j = -H x_j \end{cases}$$

$$\underline{\dot{w}} = \mathbf{J} \nabla H$$

$$\dot{z}_j = -i H \bar{z}_j$$

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbb{I}_{n \times 1} \\ -\mathbb{I}_{n \times n} & 0 \end{bmatrix}$$

(standard)

(vector)

(complex)

$$H(\underline{x}, \underline{y}) = \sum_{j=1}^n j^2 (x_j^2 + y_j^2) = \sum_{j=1}^n j^2 z_j \bar{z}_j$$

$$\dot{x}_j = (2j^2) y_j$$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} =$$

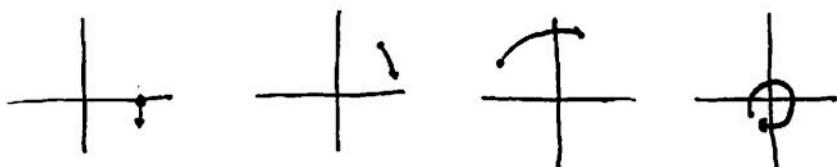
$$\dot{z}_j = -i j^2 z_j$$

$$\dot{y}_j = -(2j^2) x_j$$

(standard)

$$\text{solution: } \underline{z}_0 \mapsto \underline{z}(t)$$

$$\underline{z}_j(t) = e^{-i j^2 t} \underline{z}_0$$



$j=1$

$j=2$

$j=3$

$j=4$

$j=5$

Now consider

$H: \overbrace{\{ \text{smooth functions on } S^1 \}} \times \overbrace{\{ \text{smooth functions on } S^1 \}} \rightarrow \mathbb{R}$ .

e.g.  $x \in S^1, v: S^1 \rightarrow \mathbb{C} \text{ smooth}$ .

$$H[v, \bar{v}] = \int_{S^1} (\partial_x v)(\partial_x \bar{v}) dx$$

Hamilton Flow:  $t \mapsto v(t) \in \{ \text{smooth functions on } S^1 \}$ .

$$\dot{v} = -i H_{\bar{v}}. \quad \text{What is } H_{\bar{v}}?$$

$$\begin{aligned} \langle H_{\bar{v}}, w \rangle &:= \lim_{\varepsilon \rightarrow 0} \frac{H[v, \bar{v} + \varepsilon \bar{w}] - H[v, \bar{v}]}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{S^1} [\partial_x v \partial_x (\bar{v} + \varepsilon \bar{w}) - \partial_x v \partial_x \bar{v}] dx \\ &= \int_{S^1} \partial_x v \underbrace{\partial_x \bar{w}}_{\text{IBP}} dx = \int_{S^1} (-\partial_x^2 v) \bar{w} dx \\ &= \underbrace{\langle (-\partial_x^2 v), \bar{w} \rangle}_{\text{Directional derivative}} \end{aligned}$$

$$\implies \dot{v} = -i H_{\bar{v}} \quad \text{becomes} \quad \dot{v} = -i (-\partial_x^2 v)$$

$$\boxed{i \frac{dv}{dt} + \partial_x^2 v = 0}$$

Schrödinger's Eq.

Fourier Series

$$v: \mathbb{R}_t \times S^1_x \longrightarrow \mathbb{C}$$

↑  
periodic

$$v(t, x) = \sum_{n \in \mathbb{Z}} a_n(t) e^{inx}$$

Schrödinger Flow  $\xrightarrow{\text{regress on } t \mapsto a_n(t)}$

$$\begin{aligned} H[v, \bar{v}] &= \int_{S^1} \partial_x \left( \sum_{n \in \mathbb{Z}} a_n(t) e^{inx} \right) \partial_x \overline{\left( \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx} \right)} dx \\ &= \int_{S^1} \left( \sum_{n \in \mathbb{Z}} i n a_n(t) e^{inx} \right) \left( \sum_{k \in \mathbb{Z}} \overline{a_k(t)} (-ik) e^{-ikx} \right) dx \\ &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (in)(-ik) a_n(t) \overline{a_k(t)} \underbrace{\int_{S^1} e^{i(n-k)x} dx}_{\begin{cases} = 0 & \text{unless } n=k \\ = 1 & \text{otherwise.} \end{cases}} \\ &= \sum_{n \in \mathbb{Z}} n^2 |a_n(t)|^2 \end{aligned}$$

Thus, the linear Schrödinger flow is just the jacked up example

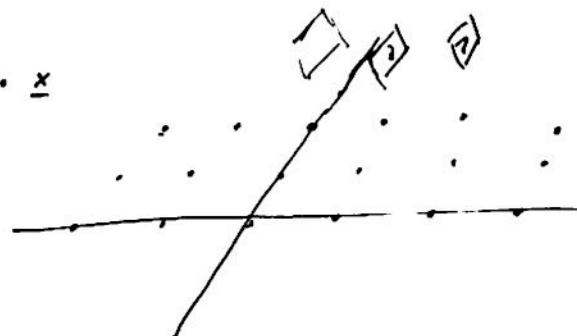


We have infinitely many decoupled Harmonic oscillators.

We can generalize these constructions:

$$v: \mathbb{R}_t \times \underbrace{[S_{x_1}^1 \times S_{x_2}^1 \times \dots \times S_{x_d}^1]}_{\mathbb{T}_{\underline{x}}^d} \rightarrow \mathbb{C}$$

$$v(t, \underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^d} a_{\underline{n}}(t) e^{i \underline{n} \cdot \underline{x}}$$



$$v: \mathbb{R}_t \times \mathbb{R}_{\underline{x}}^d \rightarrow \mathbb{C}.$$

$$v(t, \underline{x}) = \int \widehat{v(t)}(\underline{\xi}) e^{i \underline{\xi} \cdot \underline{x}} d\underline{\xi}.$$


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For the linear Schrödinger flow, we have

$$|a_{\underline{n}}(t)| = |a_{\underline{n}}(0)| \quad \text{for all time } t.$$

$$|\widehat{v(t)}(\underline{\xi})| = |\widehat{v(0)}(\underline{\xi})|$$

This means that the spatial oscillation properties of  $v(t, \underline{x})$  forever resembles those of  $v(0, \underline{x})$ .

$|a_{\underline{n}}(t)|$  frozen forever in time.



## Nonlinear Coupling

Many physical systems have wave phenomena described by the cubic nonlinear Schrödinger equation:

$$\begin{aligned} \text{NLS}_3^{\pm} & \left\{ \begin{array}{l} i\partial_t u + \Delta u = \pm |u|^2 u \\ u(0, x) = u_0(x) \end{array} \right. \iff \left\{ \begin{array}{l} \dot{u} = -i H_{\bar{u}} \\ u(0) = u_0 \end{array} \right. . \end{aligned}$$

This is also Hamiltonian:

$$H[u, \bar{u}] = \int |\nabla u(t)|^2 \pm \frac{1}{2} |u(t)|^4 dx$$

$$H_{\bar{u}} = -\Delta u \pm |u|^2 u. \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

The  $|u|^2 u$  nonlinearity couples the  $a_n(t)$  dynamics.  
 What happens? This question and other closely related questions  
 has been the focus of much of my research to date.

## Two infinite dimensional phenomena.

Weak Turbulence

$$\text{NLS}_3^+(\mathbb{T}^2).$$

Singularity Formation

$$\text{NLS}_3^-(\mathbb{R}^2).$$